

ON THE LOCAL BEHAVIOR OF $\Psi(x, y)$

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ABSTRACT. $\Psi(x, y)$ denotes the number of positive integers $\leq x$ and free of prime factors $> y$. In the range $y \geq \exp((\log \log x)^{5/3+\varepsilon})$, $\Psi(x, y)$ can be well approximated by a “smooth” function, but for $y \leq (\log x)^{2-\varepsilon}$, this is no longer the case, since then the influence of irregularities in the distribution of primes becomes apparent. We show that $\Psi(x, y)$ behaves “locally” more regular by giving a sharp estimate for $\Psi(cx, y)/\Psi(x, y)$, valid in the range $x \geq y \geq 4 \log x$, $1 \leq c \leq y$.

Introduction. The function $\Psi(x, y)$ denotes the number of positive integers $\leq x$ having no prime factors $> y$. Many arithmetic problems require accurate estimates for $\Psi(x, y)$, and the study of this function has been the object of numerous articles. A survey of the most important results as well as an extensive bibliography can be found in Norton’s memoir [8].

It turns out that $\Psi(x, y)/x$, i.e. the “probability” for a positive integer $\leq x$ to be free of prime factors $> y$, depends essentially on the ratio $\log x / \log y$. Dickman [4] showed, that for every fixed $u > 0$, the limit $\lim_{y \rightarrow \infty} \Psi(y^u, y)/y^u$ exists and equals $\rho(u)$, where ρ (the “Dickman function”) is defined by

$$\begin{aligned} \rho(u) &= 1 \quad (0 \leq u \leq 1), \quad \rho \text{ continuous at } 1, \\ \rho'(u) &= -\frac{\rho(u-1)}{u} \quad (u > 1). \end{aligned}$$

De Bruijn [3] established the quantitative estimate

$$(1) \quad \Psi(y^u, y) = y^u \rho(u) \left(1 + O_\varepsilon \left(\frac{\log(u+1)}{\log y} \right) \right)$$

for the range $y \geq 2$, $1 \leq u \leq (\log y)^{3/5-\varepsilon}$, where ε is any fixed positive number. In [7], this range was extended to $y \geq 2$, $1 \leq u \leq \exp(\log y)^{3/5-\varepsilon}$.

It is natural to ask whether these results can be further improved and, in particular, for what range relation (1) can possibly hold. The upper bound $u \leq \exp(\log y)^{3/5-\varepsilon}$ in the last mentioned result arises from the sharpest known form of the prime number theorem and could be replaced by $u \leq y^{1/2-\varepsilon}$, if the Riemann hypothesis is assumed. However, one cannot go much further, and it seems likely that for $u \approx \sqrt{y}$ relation (1) no longer holds. In fact, it appears that for $u \geq y^{1/2+\varepsilon}$, $\Psi(y^u, y)$ becomes strongly dependent on the irregularities in the distribution of primes and cannot be well approximated by “smooth” functions. In Theorem 3 we shall give an estimate, which clearly exhibits the connection between $\Psi(y^u, y)$ and the distribution properties of the prime numbers.

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The “local” behavior of $\Psi(x, y)$, i.e. the behavior of $\Psi(x', y)/\Psi(x, y)$, where x' is, in some sense, “close” to x , turns out to be more regular and can be quite accurately determined, even in a range where no good approximation for $\Psi(x, y)$ is available. This has been first noticed by Hensley, who showed recently [6, formula (11.5)] that the relation

$$\Psi(cx, y) = c^{1-\theta} \left(1 + O \left(\left(\frac{\log y}{\log x} \right)^{1/7} \right) \right)$$

holds, with a suitable $\theta = \theta(x, y)$, uniformly in the range

$$(\log x)^4 \leq y \leq \exp((\log x)^{3/4}), \quad 1 \leq c \leq 2.$$

The lower bound $y \geq (\log x)^4$ appears to be the limit of Hensley’s method. Our main object is to prove, by a completely different method, a result of the same type, but for the range $y \geq 4 \log x$.

We first derive an upper bound for $\Psi(cx, y)$ in the form of an *exact* inequality, which is valid, whenever y is sufficiently large, $x \geq y$ and c not too close to 1, without any further condition on the relative size of x and y .

THEOREM 1. *The inequality $\Psi(cx, y) \leq c\Psi(x, y)$ holds for all x, y and c satisfying $x \geq y \geq y_0$, $c \geq 1 + \exp(-\sqrt{\log y})$, where y_0 is an absolute constant ≥ 2 .*

In our second and main theorem we give a sharp estimate for $\Psi(cx, y)/\Psi(x, y)$, which is essentially the same as Hensley’s estimate with a slightly weaker error term, but valid in a substantially larger range.

THEOREM 2. *Uniformly for $x \geq e^4$, $4 \log x \leq y \leq x$, $1 \leq c \leq y$, we have*

$$\Psi(cx, y) = c^{\alpha(x, y)} \Psi(x, y) \left(1 + O \left(\left(\frac{\log y}{\log x} \right)^{1/10} \right) \right),$$

where $\alpha(x, y)$ is the (unique) real number satisfying

$$\sum_{p \leq y} \frac{\log p}{p^{\alpha(x, y)} - 1} = \log x.$$

By iterating the estimate of Theorem 2, one can obtain estimates for $\Psi(cx, y)/\Psi(x, y)$ in the case $c > y$. One could even deduce an estimate for $\Psi(x, y)$ itself, but this would be much weaker than that given in Theorem 3.

If x is large compared with y , then the estimate of Theorem 2 is surprisingly sharp. For example, in the case $x = \exp(y^\gamma)$, where $\gamma \in (0, 1)$ is fixed, the error term is of order $O((y^{-\gamma} \log y)^{1/10})$ and hence considerably smaller than the error term in (1). For small x , $x \leq \exp((\log y)^{20})$ say, the estimate of the theorem becomes less effective, but in this range (1) can be applied.

The quantity $\alpha(x, y)$ is defined by a prime number sum and depends on the behavior of the error term in the prime number theorem. It is not difficult to replace in the estimate of Theorem 2 $\alpha(x, y)$ by a smooth function, at the cost of a weaker error term. Indeed, one can show (cf. Lemma 4) that, under the hypotheses of Theorem 2, we have

$$\alpha(x, y) = 1 - \frac{\xi_u}{\log y} + O \left(\frac{1}{u \log y} \right) + O \left(\frac{u}{y} \right) + O(\exp(-\sqrt{\log y})),$$

where $u = \log x / \log y$, and ξ_u is the positive solution of $e^{\xi_u} = u\xi_u + 1$. According to [1, Lemma 3], the estimate

$$\frac{\rho(u+t)}{\rho(u)} = e^{t\xi_{u+t}} \left(1 + O\left(\frac{1}{u}\right)\right) = e^{t\xi_u} \left(1 + O\left(\frac{1}{u}\right)\right)$$

holds uniformly for $u \geq 1$ and $0 \leq t \leq 1$. Thus we see that, for $1 \leq c \leq y$,

$$c^{\xi_u / \log y} = \frac{\rho(u + (\log c)/(\log y))}{\rho(u)} \left(1 + O\left(\frac{1}{u}\right)\right),$$

and after a slight change in notation, we arrive at the following corollary to Theorem 2:

COROLLARY. *Uniformly for $y \geq 2$, $1 \leq u \leq y/4 \log y$, $u \leq u' \leq u + 1$ we have*

$$\begin{aligned} \frac{\Psi(y^{u'}, y)}{\Psi(y^u, y)} &= \frac{y^{u'} \rho(u')}{y^u \rho(u)} \left(1 + O\left(\frac{1}{u^{1/10}}\right) + O\left(\frac{u(u' - u) \log y}{y}\right) \right. \\ &\quad \left. + O\left(\exp\left(-\frac{1}{2}\sqrt{\log y}\right)\right)\right). \end{aligned}$$

This result shows that in the range $1 \leq u \leq y/4 \log y$, $\Psi(y^u, y)$ behaves “locally” like $y^u \rho(u)$, although the relation $\Psi(y^u, y) \sim y^u \rho(u)$ is known to be valid only for a much smaller range.

The proof of Theorems 1 and 2 is based on the identity

$$(2) \quad \Psi(x, y) \log x = \int_1^x \frac{\Psi(t, y)}{t} dt + \sum_{\substack{p^m \leq x \\ p \leq y}} \Psi\left(\frac{x}{p^m}, y\right) \log p \quad (x \geq y \geq 2),$$

which can be easily established starting from the weighted sum $\sum \log n$, where the summation runs over all positive integers $n \leq x$, which are free of prime factors $> y$ (cf. [7]).

By a repeated application of this identity, one can reduce an upper bound for $\sup_{x \geq y} (\Psi(cx, y)/\Psi(x, y))$ to an upper bound for $\sup_{y \leq x \leq y^2} (\Psi(cx, y)/\Psi(x, y))$, which is not difficult to establish. This method leads in a fairly simple way to the assertion of Theorem 1.

The proof of Theorem 2 is much more complicated. The main reason for this lies in the fact that the error term in the asserted estimate decreases with x , when y is fixed. The method of reducing such an estimate by means of the identity (2) to the case of small x , can no longer be applied, since the error term would be relatively poor for small x and would certainly not decrease, when x gets larger. Instead of “descending” down to small values of x , we shall therefore descend from a given x only down to a certain x' , which will be of order $x \exp(-(\log x)^\gamma)$ for a suitable $\gamma \in (0, 1)$. Using an idea of de Bruijn [2], we shall then show that the error term in the desired formula for x must be substantially smaller than the error term for x' , if it is not very small itself. To obtain the required estimate, a relatively crude estimate of the error term for x' will then be sufficient. This is achieved by the following estimate for $\Psi(x, y)$, which is of interest by itself.

THEOREM 3. *Uniformly for $x \geq e^4$, $2 \log x \leq y \leq x$ we have*

$$\Psi(x, y) = \exp\{I(x, y) + O((\log y)^5)\},$$

where

$$I(x, y) = \int_y^x \frac{\alpha(t, y)}{t} dt.$$

This result may be compared with the sharpest previously known estimate for $\Psi(x, y)$ in the range $x \geq 3$, $(\log x)^{1+\varepsilon} \leq y \leq x$, namely (cf. [7, Theorem 2] and the subsequent remarks)

$$(3) \quad \log \Psi(x, y) = \log(x\rho(u)) + O_\varepsilon(u \exp(-(\log u)^{3/5-\varepsilon})) \quad (u = \log x / \log y).$$

(3) could be easily deduced from Theorem 3 in the case $u \geq \exp \sqrt{\log y}$, and follows from (1), if $1 \leq u < \exp \sqrt{\log y}$. For large u , the error term in Theorem 3 is considerably smaller than the error term in (3). We have to however keep in mind that the function $I(x, y)$ is not “smooth” and depends on irregularities in the distribution of primes, whereas in (3) the influence of such irregularities has to be absorbed by the error term. Due to these irregularities, the function $I(x, y)$ “oscillates” around a smooth approximation $I_0(x, y)$, and it can be shown that for $u = \log x / \log y > y^{1/2+\varepsilon}$ these oscillations are of larger order than the error term in Theorem 3. Thus Theorem 3 justifies, in a sense, our earlier assertion, that for $u > y^{1/2+\varepsilon}$, $\Psi(y^u, y)$ cannot be approximated too well by a smooth function.

2. Proof of Theorem 1. We first remark that it suffices to prove the theorem in the case $c \leq 4$; the extension to $c > 4$ follows by iteration.

Put, for $c \geq 1$, $y \geq 2$ and $u \geq 1$,

$$q(c, y; u) = \Psi(cy^u, y) / \Psi(y^u, y)$$

and

$$q^*(c, y; u) = \sup\{q(c, y; u') : 1 \leq u' \leq u\}.$$

We have to show that for $y \geq y_0$, $1 + \exp(-\sqrt{\log y}) \leq c \leq 4$, $u \geq 1$, we have $q^*(c, y; u) \leq c$. We shall prove this in two steps, given by the following two lemmas:

LEMMA 1. *If y_0 is sufficiently large and $y \geq y_0$, $1 + y^{-1/3} \leq c \leq 4$, then we have $q^*(c, y; u) = q^*(c, y; 2)$ for every $u \geq 2$.*

LEMMA 2. *If y_0 is sufficiently large and $y \geq y_0$, $1 + \exp(-\sqrt{\log y}) \leq c \leq 4$, then we have $q^*(c, y; 2) \leq c$.*

PROOF OF LEMMA 1. Applying the identity (2) twice on taking $x = cy^u$ and $x = y^u$, as well as the inequality

$$\Psi(ct, y) \leq q^*(c, y; u) \Psi(t, y) \quad (y \leq t \leq y^u),$$

we get, for $c \geq 1$, $y \geq 2$ and $u \geq 2$,

$$\begin{aligned} q(c, y; u) \Psi(y^u, y) \log(cy^u) &= \Psi(cy^u, y) \log(cy^u) \\ &= \int_1^{cy^u} \frac{\Psi(t, y)}{t} dt + \sum_{\substack{p^m \leq cy^u \\ p \leq y}} \Psi\left(\frac{cy^u}{p^m}, y\right) \log p \\ &= \int_{1/c}^{y^u} \frac{\Psi(ct, y)}{t} dt + \sum_{\substack{p^m \leq cy^u \\ p \leq y}} \Psi\left(\frac{cy^u}{p^m}, y\right) \log p \\ &\leq q^*(c, y; u) \left\{ \int_y^{y^u} \frac{\Psi(t, y)}{t} dt + \sum_{\substack{p^m \leq y^{u-1} \\ p \leq y}} \Psi\left(\frac{y^u}{p^m}, y\right) \log p \right\} + R(c, y; u) \\ &\leq q^*(c, y; u) \Psi(y^u, y) \log y^u + R(c, y; u), \end{aligned}$$

where

$$R(c, y; u) = \int_{1/c}^y \frac{\Psi(ct, y)}{t} dt + \sum_{\substack{y^{u-1} < p^m \leq cy^u \\ p \leq y}} \Psi\left(\frac{cy^u}{p^m}, y\right) \log p.$$

In view of the trivial estimate $q(c, y; u) \geq 1$, we therefore obtain

$$q(c, y; u) \leq q^*(c, y; u) + \frac{R(c, y; u)}{\Psi(y^u, y) \log y^u} - \frac{\log c}{\log y^u}.$$

We shall presently show that the estimate

$$(4) \quad R(c, y; u) \ll \Psi(y^u, y) \frac{\log y}{y^{1/2}}$$

holds uniformly in the range $1 \leq c \leq 4$, $y \geq 2$, $u \geq 2$. (4) implies that for $y \geq y_0$, $1 + y^{-1/3} \leq c \leq 4$, $u \geq 2$, the quantity $R(c, y; u)/\Psi(y^u, y) - \log c$ is negative and hence $q(c, y; u) < q^*(c, y; u)$. This means that the function $u \mapsto q^*(c, y; u)$ is nonincreasing and therefore constant for $u \geq 2$, as we wanted to show.

It remains to prove (4). From the trivial estimate $\Psi(t, y) \leq t$ and our assumption $1 \leq c \leq 4$ we get

$$\begin{aligned} R(c, y; u) &\leq \int_{1/c}^y c dt + \sum_{\substack{y^{u-1} < p^m \leq cy^u \\ p \leq y}} \left(\frac{cy^u}{p^m}\right) \log p \\ &\leq 4y + 4y^u (\log y) \sum_{p \leq y} \sum_{\substack{m \geq 1 \\ p^m > y^{u-1}}} \frac{1}{p^m}. \end{aligned}$$

The double sum can be estimated by

$$\ll \sum_{p \leq y} \frac{1}{y^{u-1}} \ll \frac{y^{2-u}}{\log y},$$

but also by

$$\begin{aligned} &\leq \sum_{p \leq \sqrt{y}} \sum_{\substack{m \geq 1 \\ p^m > y^{u-1}}} \frac{1}{p^m} + \sum_{\sqrt{y} < p \leq y} \sum_{m \geq 2} \frac{1}{p^m} \\ &\ll \frac{y^{3/2-u}}{\log y} + y^{-1/2} \ll y^{-1/2}, \end{aligned}$$

since $u \geq 2$. Hence we obtain

$$\begin{aligned} R(c, y; u) &\ll y + y^u \min(y^{2-u}, y^{-1/2} \log y) \\ &\ll \min(y^2, y^{u-1/2} \log y). \end{aligned}$$

This together with the crude estimate

$$\Psi(y^u, y) \begin{cases} \gg y^u & (1 \leq u \leq 5/2), \\ \geq \Psi(y^{5/2}, y) \gg y^{5/2} & (u > 5/2) \end{cases}$$

yields (4) and thus completes the proof of Lemma 1.

PROOF OF LEMMA 2. By the inclusion-exclusion principle we have, for $y \leq x < y^2$ and $1 \leq c \leq 4$,

$$\Psi(x, y) = [x] - \sum_{y < p \leq x} \left[\frac{x}{p} \right]$$

and

$$\Psi(cx, y) = [cx] - \sum_{y < p \leq cx} \left[\frac{cx}{p} \right] + O \left(x \left(\sum_{y < p \leq cy} \frac{1}{p} \right)^2 \right).$$

A sharp form of the prime number theorem shows that the error term in the second formula is of order

$$O \left(x \frac{(c-1)^2}{\log^2 y} \right) + O(x \exp(-2\sqrt{\log y})).$$

Moreover, we have

$$\begin{aligned} \sum_{y < p \leq x} \left[\frac{x}{p} \right] &= \sum_{y < p \leq x} \sum_{n \leq x/p} 1 = \sum_{n \leq x} \sum_{y < p \leq x/n} 1 \\ &= \sum_{n \leq x/y} \int_y^{x/n} \frac{dt}{\log t} + O(x \exp(-2\sqrt{\log y})) \\ &= \int_y^x \left[\frac{x}{t} \right] \frac{dt}{\log t} + O(x \exp(-2\sqrt{\log y})), \end{aligned}$$

and similarly

$$\begin{aligned} \sum_{y < p \leq cx} \left[\frac{cx}{p} \right] &= \int_y^{cx} \left[\frac{cx}{t} \right] \frac{dt}{\log t} + O(x \exp(-2\sqrt{\log y})) \\ &= c \int_{y/c}^x \left[\frac{x}{t} \right] \frac{dt}{\log(ct)} + O(x \exp(-2\sqrt{\log y})). \end{aligned}$$

Thus we see that

$$\Psi(cx, y) - c\Psi(x, y) = c(I_1 - I_2) + O \left(x \frac{(c-1)^2}{\log^2 y} \right) + O(x \exp(-2\sqrt{\log y})),$$

where

$$I_1 = \int_y^x \left[\frac{x}{t} \right] \left(\frac{1}{\log t} - \frac{1}{\log(ct)} \right) dt$$

and

$$I_2 = \int_{y/c}^y \left[\frac{x}{t} \right] \frac{dt}{\log(ct)}.$$

We have

$$\begin{aligned} I_1 &\leq x(\log c) \int_y^x \frac{dt}{t \log^2 t} \\ &= x(\log c) \left(\frac{1}{\log y} - \frac{1}{\log x} \right) \leq \frac{x}{2} \cdot \frac{\log c}{\log y} \end{aligned}$$

and

$$\begin{aligned} I_2 &\geq \frac{1}{\log(yc)} \int_{y/c}^y \left(\frac{x}{t} - 1 \right) dt \\ &= \frac{1}{\log(yc)} \left(x \log c - y + \frac{y}{c} \right) \\ &\geq \frac{x \log c}{\log y} \left(1 - \frac{y}{x} \right) + O \left(x \frac{\log c}{\log^2 y} \right). \end{aligned}$$

If now $x \geq 3y$, then

$$I_1 - I_2 \leq -\frac{x \log c}{6 \log y} \left(1 + O \left(\frac{1}{\log y} \right) \right)$$

and we obtain $\Psi(cx, y) \leq c\Psi(x, y)$, provided $y \geq y_0$, $1 + \exp(-\sqrt{\log y}) \leq c \leq 4$ and y_0 is sufficiently large. If $y \leq x < 3y$, then we have

$$\begin{aligned} I_1 &= O \left(x \frac{\log c}{\log^2 y} \right), \\ I_2 &\geq \int_{y/c}^y \frac{dt}{\log(ct)} \geq \frac{y}{\log(cy)} \left(1 - \frac{1}{c} \right) \\ &\geq \frac{x}{3} \cdot \frac{\log c}{\log(cy)} = \frac{x}{3} \cdot \frac{\log c}{\log y} + O \left(x \frac{\log c}{\log^2 y} \right), \end{aligned}$$

and under the above hypotheses on c and y , we get again the inequality $\Psi(cx, y) \leq c\Psi(x, y)$. Thus we have proved (under the stated hypotheses)

$$(q^*(c, y; 2) =) \sup_{y \leq x \leq y^2} \frac{\Psi(cx, y)}{\Psi(x, y)} \leq c,$$

i.e. the assertion of Lemma 2. The proof of Theorem 1 is now complete.

For later use we state the following lemma, which is an easy consequence of Theorem 1 and Lemma 1.

LEMMA 3. *The estimate*

$$\Psi(cx, y) = \Psi(x, y)(1 + O(c - 1) + O(y^{-1/3}))$$

holds for all $c \geq 1$ and $x \geq y \geq y_0$, where y_0 is a sufficiently large constant.

PROOF. The lower estimate is trivial, since the function $x \mapsto \Psi(x, y)$ is non-decreasing. In the case $c \geq 1 + \exp(-\sqrt{\log y})$, the upper estimate follows from

Theorem 1. Moreover, if $1 + y^{-1/3} \leq c < 1 + \exp(-\sqrt{\log y})$, then Lemma 1 together with the trivial estimate

$$\begin{aligned} q^*(c, y; 2) &= \sup_{1 \leq u \leq 2} \frac{\Psi(cy^u, y)}{\Psi(y^u, y)} \\ &\leq \sup_{1 \leq u \leq 2} \frac{1}{\Psi(y^u, y)} \left(\Psi(y^u, y) + \sum_{y^u < n \leq cy^u} 1 \right) \\ &\leq \sup_{1 \leq u \leq 2} \left(1 + \frac{(c-1)y^u + 1}{\Psi(y^u, y)} \right) \\ &= 1 + O(c-1) + O\left(\frac{1}{y}\right) \end{aligned}$$

yields the desired upper estimate. Finally, if $1 \leq c \leq 1 + y^{-1/3}$, the result follows from the inequality $\Psi(cx, y) \leq \Psi((1 + y^{-1/3})x, y)$ and the already proved cases.

3. Proofs of Theorems 2 and 3. Beginning. We shall consider the variable y as a parameter and keep it fixed throughout the proof. We may suppose for the proofs of both theorems y to be sufficiently large. All estimates in the sequel are understood to hold uniformly for all sufficiently large y . With these conventions, we shall, in general, not indicate an eventual dependency on y in the notation; in particular, we put $\alpha_u = \alpha(y^u, y)$.

We start with some simple estimates for the quantities ξ_u and $\alpha_u = \alpha(y^u, y)$. Recall that ξ_u is defined by the equation $e^{\xi_u} = u\xi_u + 1$.

LEMMA 4. *We have*

- (i) $\xi_u \sim \log u \quad (u \rightarrow \infty),$
- (ii) $\frac{d}{du} \xi_u = \frac{1}{u} \left(1 + O\left(\frac{1}{\log u}\right) \right) \quad (u \geq 2),$
- (iii) $0 \leq \frac{d}{du} \alpha_u \ll \frac{1}{u} \quad \left(1 \leq u \leq \frac{y}{2 \log y} \right),$
- (iv) $\alpha_u = 1 + O\left(\frac{1}{\log y}\right) \quad (1 \leq u \leq 2),$
 $\frac{\log 3}{\log y} + O\left(\frac{1}{\log^2 y}\right) \leq \alpha_u$
 $\leq 1 - \frac{\log 2}{\log y} + O\left(\frac{1}{\log^2 y}\right) \quad \left(2 \leq u \leq 2 \frac{y}{2 \log y} \right),$
- (v) $\frac{1}{u \log y} \sum_{\substack{p^m > y^t \\ p \leq y}} \frac{\log p}{p^{m\alpha_u}} \leq e^{1-t}$
 $(y \text{ sufficiently large, } 1 \leq u \leq y/2 \log y, t \geq 1),$

$$(vi) \quad \alpha_u = 1 - \frac{\xi_u}{\log y} + O\left(\frac{1}{u \log y}\right) + O\left(\frac{u}{y}\right) + O(\exp(-\sqrt{\log y}))$$

$$(1 \leq u \leq y/2 \log y).$$

PROOF. For sufficiently large u we have

$$e^{\log u} < u \log u + 1, \quad e^{\log(u \log^2 u)} > u \log(u \log^2 u) + 1,$$

so that

$$\log u < \xi_u < \log(u \log^2 u).$$

This yields the relation $\xi_u \sim \log u$ ($u \rightarrow \infty$), stated in part (i) of the lemma. Taking the derivative on both sides of the equation defining ξ_u , we find

$$e^{\xi_u} \frac{d}{du} \xi_u = \xi_u + u \frac{d}{du} \xi_u,$$

whence, for $u \geq 2$,

$$\frac{d}{du} \xi_u = \frac{\xi_u}{e^{\xi_u} - u} = \frac{\xi_u}{u(\xi_u - 1) + 1} = \frac{1}{u} \left(1 + O\left(\frac{1}{\log u}\right)\right),$$

which is estimate (ii) of the lemma.

Differentiating both sides of the identity

$$\log y^u = \sum_{p \leq y} \frac{\log p}{p^{\alpha_u} - 1}$$

with respect to u , we get

$$\log y = \left(\sum_{p \leq y} \frac{p^{\alpha_u} \log^2 p}{(p^{\alpha_u} - 1)^2} \right) \frac{d}{du} \alpha_u,$$

and hence

$$0 \leq \frac{d}{du} \alpha_u \leq (\log y) \left((\log 2) \sum_{p \leq y} \frac{\log p}{p^{\alpha_u} - 1} \right)^{-1}$$

$$= \frac{(\log 2)^{-1}}{u} \ll \frac{1}{u},$$

i.e. part (iii) of the lemma.

The lower estimate

$$\alpha_u \geq \frac{\log 3}{\log y} + O\left(\frac{1}{\log^2 y}\right) \quad \left(1 \leq u \leq \frac{y}{2 \log y}\right)$$

follows from the inequality

$$\frac{y}{2} \geq u \log y = \sum_{p \leq y} \frac{\log p}{p^{\alpha_u} - 1}$$

$$\geq \frac{1}{y^{\alpha_u} - 1} \sum_{p \leq y} \log p = \frac{y}{y^{\alpha_u} - 1} \left(1 + O\left(\frac{1}{\log y}\right)\right),$$

and a similar argument yields the other estimates in part (iv).

(v) follows from the inequality

$$\frac{1}{u \log y} \sum_{p \leq y} \sum_{\substack{m \geq 1 \\ p^m > y^t}} \frac{1}{p^{m\alpha_u}} \leq \frac{1}{y^{(t-1)\alpha_u}} \cdot \frac{1}{u \log y} \sum_{p \leq y} \frac{\log p}{p^{\alpha_u} - 1} = y^{(1-t)\alpha_u}$$

and the lower bound for α_u given in part (iv) of the lemma.

It remains to prove formula (vi). This formula holds trivially, if $1 \leq u \leq 2$, since in this range ξ_u is bounded and, by part (iv) of the lemma, $1 - \alpha_u = O(1/\log y)$. If $2 \leq u \leq y/2 \log y$ and y is sufficiently large, as we may assume, then we have, by the second estimate of part (iv),

$$\frac{1}{\log y} \leq \alpha_u \leq 1 - \frac{1}{2 \log y}.$$

In view of this inequality, we readily obtain, by partial summation and a strong form of the prime number theorem,

$$\begin{aligned} u \log y &= \sum_{p \leq y} \frac{\log p}{p^{\alpha_u} - 1} = (1 + O(\exp(-\sqrt{\log y}))) \int_2^y \frac{dt}{t^{\alpha_u} - 1} + O(1) \\ &= \left(1 + O\left(\frac{1}{u \log y}\right) + O(\exp(-\sqrt{\log y}))\right) \int_2^y \frac{dt}{t^{\alpha_u} - 1} \\ &= \left(1 + O\left(\frac{1}{u \log y}\right) + O(\exp(-\sqrt{\log y})) + O(y^{-\alpha_u}) + O(y^{\alpha_u-1})\right) \frac{y^{1-\alpha_u}}{1-\alpha_u}. \end{aligned}$$

The last two error terms can be estimated by

$$y^{-\alpha_u} \ll \frac{1-\alpha_u}{y} \int_2^y \frac{dt}{t^{\alpha_u} - 1} \ll \frac{u \log y}{y}$$

and

$$y^{\alpha_u-1} \ll \frac{1}{1-\alpha_u} \left(\int_2^y \frac{dt}{t^{\alpha_u} - 1}\right)^{-1} \ll \frac{1}{u}.$$

Hence we get

$$\frac{y^{1-\alpha_u}}{1-\alpha_u} \ll u \log y = \frac{y^{1-\alpha_u}}{1-\alpha_u} \left(1 + O\left(\frac{1}{u}\right) + O\left(\frac{u \log y}{y}\right) + O(\exp(-\sqrt{\log y}))\right).$$

Putting

$$u_1 = \frac{y^{1-\alpha_u} - 1}{(1-\alpha_u) \log y},$$

we have, by the definition of ξ_{u_1} ,

$$1 - \alpha_u = \xi_{u_1} / \log y,$$

and the above estimate implies

$$1 \ll \frac{u}{u_1} = 1 + O\left(\frac{1}{u}\right) + O\left(\frac{u \log y}{y}\right) + O(\exp(-\sqrt{\log y})).$$

In view of part (ii) of the lemma, we therefore obtain

$$\begin{aligned} 1 - \alpha_u - \frac{\xi_u}{\log y} &= \frac{\xi_{u_1} - \xi_u}{\log y} \ll \frac{|u - u_1| + 1}{u \log y} \\ &= O\left(\frac{1}{u \log y}\right) + O\left(\frac{u}{y}\right) + O(\exp(-\sqrt{\log y})), \end{aligned}$$

i.e. the desired formula. This completes the proof of Lemma 4.

Define, for $u \geq 1$, $f(u)$ by

$$\Psi(y^u, y) = f(u) \exp \left((\log y) \int_1^u \alpha_t dt \right).$$

Theorems 2 and 3 can be easily reformulated in terms of $f(u)$: Theorem 3 is equivalent to the estimate

$$f(u) = \exp\{O((\log y)^5)\} \quad (1 \leq u \leq y/2 \log y),$$

and Theorem 2 roughly says that the function $f(u)$ does not change too rapidly. The starting point for the proof of both results is an approximate functional equation for $f(u)$, given by the following

LEMMA 5. Put $h = 10 \log y$ and suppose y is sufficiently large. Then we have, for $h^3 \leq u \leq y/2 \log y$,

$$\left(1 + O\left(\frac{h^3}{u}\right)\right) f(u) \log y^u = \sum_{\substack{p^m \leq y^h \\ p \leq y}} \frac{\log p}{p^{m\alpha_u}} f\left(u - \frac{\log p^m}{\log y}\right).$$

PROOF. The proof is based on the identity (2), which can be written as

$$\Psi(y^u, y) \log y^u = \int_1^{y^u} \frac{\Psi(t, y)}{t} dt + \sum_{\substack{p^m \leq y^u \\ p \leq y}} \Psi\left(\frac{y^u}{p^m}, y\right) \log p \quad (u \geq 1).$$

We shall show that for $h^3 \leq u \leq y/2 \log y$ the main contribution to the right-hand side comes from the part of the sum corresponding to $p^m \leq y^h$.

The above identity implies

$$\begin{aligned} \Psi(y^u, y) \log y^u &\geq \sum_{p \leq y} \Psi\left(\frac{y^u}{p}, y\right) \log p \\ &\geq \Psi(y^{u-1}, y) \sum_{p \leq y} \log p = \Psi(y^{u-1}, y) y \left(1 + O\left(\frac{1}{\log y}\right)\right) \end{aligned}$$

for every $u \geq 1$. If now, according to the hypothesis of the lemma, y is sufficiently large and $u \leq y/2 \log y$, it follows that

$$\Psi(y^{u-1}, y) \leq \frac{2}{3} \Psi(y^u, y),$$

and by iteration we get

$$\Psi\left(\frac{y^u}{t}, y\right) \leq \left(\frac{2}{3}\right)^{[\log t / \log y]} \Psi(y^u, y)$$

for $1 \leq t \leq y^u$. Thus we see that

$$\begin{aligned} \int_1^{y^u} \Psi(t, y) \frac{dt}{t} &= \int_1^{y^u} \Psi\left(\frac{y^u}{t}, y\right) \frac{dt}{t} \\ &= \Psi(y^u, y) \int_1^{y^u} \left(\frac{2}{3}\right)^{[\log t / \log y]} \frac{dt}{t} \ll \Psi(y^u, y) \log y. \end{aligned}$$

Moreover, we have, for $h \leq u \leq y/2 \log y$,

$$\begin{aligned}\Psi(y^{u-h}, y) &\leq \left(\frac{2}{3}\right)^{[h]} \Psi(y^u, y) \\ &= \Psi(y^u, y) \exp\left(-[10 \log y] \log\left(\frac{3}{2}\right)\right) \ll \frac{\Psi(y^u, y)}{y^2},\end{aligned}$$

so that

$$\begin{aligned}\sum_{\substack{y^h < p^m \leq y^u \\ p \leq y}} \Psi\left(\frac{y^u}{p^m}, y\right) \log p &\leq \Psi(y^{u-h}, y) \sum_{p \leq y} \sum_{\substack{m \geq 1 \\ p^m \leq y^u}} \log p \\ &\leq \Psi(y^{u-h}, y) \sum_{p \leq y} \log y^u \ll \frac{\Psi(y^u, y) \log y^u}{y}.\end{aligned}$$

We therefore arrive at the estimate

$$\left(1 + O\left(\frac{1}{u}\right)\right) \Psi(y^u, y) \log y^u = \sum_{\substack{p^m \leq y^h \\ p \leq y}} \Psi\left(\frac{y^u}{p^m}, y\right) \log p,$$

valid, whenever y is sufficiently large and $h \leq u \leq y/2 \log y$. The asserted formula now follows on noting that, by definition of $f(t)$ and estimate (iii) of Lemma 4, we have, for $h^3 \leq u \leq y/2 \log y$ and $1 \leq t \leq y^h$,

$$\begin{aligned}\frac{\Psi(y^u/t, y)}{\Psi(y^u, y)} &= \frac{f(u - (\log t)/(\log y))}{f(u)} \exp\left\{-(\log y) \int_{u - (\log t)/(\log y)}^u \alpha_s ds\right\} \\ &= \frac{f(u - (\log t)/(\log y))}{f(u)} \exp\left\{-(\log y) \int_{u - (\log t)/(\log y)}^u \left(\alpha_u + O\left(\frac{u-s}{u}\right)\right) ds\right\} \\ &= \frac{f(u - (\log t)/(\log y))}{f(u)} \cdot \frac{1}{t^{\alpha_u}} \exp\left\{O\left(\frac{h^3}{u}\right)\right\}.\end{aligned}$$

4. Proof of Theorem 3. With $f(u)$ and h being defined as before, we put, for $u \geq h+1$,

$$f^*(u) = \sup\{f(u') : u-h \leq u' \leq u\}$$

and

$$f_*(u) = \inf\{f(u') : u-h \leq u' \leq u\}.$$

In order to obtain Theorem 3, we have to estimate $f(u)$ from above and below. To this end we shall show in the next lemma that the functions $f^*(u)$ (resp. $f_*(u)$) are essentially nonincreasing (resp. nondecreasing), and then deduce the required estimate from a trivial estimate in the case of small u .

LEMMA 6. *If y is sufficiently large, $h^4 \leq u \leq y/2 \log y$ and $0 \leq t \leq h$, we have*

$$f^*(u) \leq f^*(u-t) (1 + O(h^4/u))$$

and

$$f_*(u) \geq f_*(u-t) (1 + O(h^4/u)).$$

PROOF. It suffices to prove these estimates in the case $0 \leq t \leq 1/2$, with $O(h^3/u)$ instead of $O(h^4/u)$ as error term and for the slightly extended range

$h^4 - h \leq u \leq y/2 \log y$; the general case follows by iteration. We shall only prove the second estimate; the proof of the first is similar and slightly simpler.

Let $u \in [h^4 - h, y/2 \log y]$ and $t \in [0, \frac{1}{2}]$ be fixed and let $u_1 \in [u - t, u]$ be such that

$$f(u_1) \geq (1 - 1/u) \inf\{f(u') : u - t \leq u' \leq u\}.$$

From Lemma 5 we get, assuming y to be sufficiently large,

$$\begin{aligned} \left(1 + O\left(\frac{h^3}{u}\right)\right) f(u_1) &\geq f(u_1) \cdot \frac{1}{u_1 \log y} \sum_{\substack{p^m \leq y^{u_1 - u + t} \\ p \leq y}} \frac{\log p}{p^{m\alpha_{u_1}}} \\ &\quad + f_*(u - t) \cdot \frac{1}{u_1 \log y} \sum_{\substack{y^{u_1 - u + t} < p^m \leq y^h \\ p \leq y}} \frac{\log p}{p^{m\alpha_{u_1}}} \\ &= f(u_1)\lambda + f_*(u - t)\mu, \end{aligned}$$

say. By estimate (v) of Lemma 4 and the definition of h we have

$$\lambda + \mu = \frac{1}{u_1 \log y} \sum_{\substack{p^m \leq y^h \\ p \leq y}} \frac{\log p}{p^{m\alpha_{u_1}}} = 1 + O(e^{1-h}) = 1 + O\left(\frac{1}{u}\right).$$

Moreover, using the inequality $u_1 - u + t \leq t \leq 1/2$ and the estimates for α_{u_1} from part (iv) of Lemma 4, it is easy to see that

$$\mu \geq \frac{1}{u_1 \log y} \sum_{\sqrt{y} < p \leq y} \frac{\log p}{p^{\alpha_{u_1}}} \gg \frac{1}{u_1 \log y} \sum_{p \leq y} \frac{\log p}{p^{\alpha_{u_1} - 1}} = 1,$$

the implied constant being absolute. We therefore get

$$f_*(u - t) \leq \frac{1}{\mu} \left(1 + O\left(\frac{h^3}{u}\right) - \lambda\right) f(u_1) = \left(1 + O\left(\frac{h^3}{u}\right)\right) f(u_1).$$

Since

$$f_*(u) \geq \min(f(u_1), f_*(u - t)),$$

we conclude

$$f_*(u) \geq (1 + O(h^3/u)) f_*(u - t),$$

as wanted.

PROOF OF THEOREM 3. The integral $I(x, y)$ from the statement of Theorem 3 is connected with the function $f(u)$ via the formula

$$\Psi(y^u, y) = f(u) \exp \left\{ (\log y) \int_1^u \alpha_t dt \right\} = f(u) e^{I(x, y)}.$$

Thus we have to prove the estimate

$$f(u) = \exp(O(h^5)) \quad (1 \leq u \leq y/2 \log y).$$

For $1 \leq u \leq h^4$, this holds trivially, since then

$$1 \leq \Psi(y^u, y) \leq y^u \leq \exp(h^5)$$

and

$$1 \leq \exp \left((\log y) \int_1^u \alpha_t dt \right) = \exp(O(hu)) = \exp(O(h^5)).$$

If now $h^4 < u \leq y/2 \log y$, y being sufficiently large, as we may assume, we obtain by iterating the inequalities of Lemma 6 with $t = h$,

$$\begin{aligned} f^*(u) &\leq f^*(u_0) \exp \left(O \left(h^4 \sum_{\substack{n \geq 0 \\ u - nh > h^4}} \frac{1}{u - nh} \right) \right) \\ &= f^*(u_0) \exp \left(O \left(h^4 \frac{\log u}{h} \right) \right) = f^*(u_0) \exp(O(h^4)), \end{aligned}$$

and similarly

$$f_*(u) \geq f_*(u_0) \exp(O(h^4))$$

for some u_0 satisfying $h^4 - h < u_0 \leq h^4$. By the above treated trivial case, we have

$$f^*(u_0) = \sup\{f(u') : u_0 - h \leq u' \leq u_0\} = \exp(O(h^5))$$

and

$$f_*(u_0) = \inf\{f(u') : u_0 - h \leq u' \leq u_0\} = \exp(O(h^5)).$$

We therefore get

$$f^*(u) \leq \exp(O(h^5)), \quad f_*(u) \geq \exp(O(h^5)),$$

and since $f_*(u) \leq f(u) \leq f^*(u)$, we obtain the required estimate $f(u) = \exp(O(h^5))$. The proof of Theorem 3 is now complete.

5. Proof of Theorem 2: the key lemma. We keep the notations introduced so far and define in addition, for $u_2 \geq u_1 \geq 1$,

$$f^*(u_1, u_2) = \sup\{f(u) : u_1 \leq u \leq u_2\},$$

$$f_*(u_1, u_2) = \inf\{f(u) : u_1 \leq u \leq u_2\}.$$

Thus we have, for $u \geq h + 1$,

$$f^*(u) = f^*(u - h, u), \quad f_*(u) = f_*(u - h, u).$$

Let

$$\Delta(u) = f^*(u) - f_*(u) = \sup\{f(u') - f(u'') : u - h \leq u', u'' \leq u\}.$$

The quantity $\Delta(u)$ measures the oscillations of $f(u)$ in the interval $[u - h, u]$, and we have to show that it is small compared with $f^*(u)$. To this end, we adapt an idea of de Bruijn [2], originally used to show the convergence of solutions of certain Volterra integral equations. Lemma 5 shows that $f(u)$ satisfies indeed a kind of Volterra equation. The argument, however, is more complicated than in de Bruijn's case, since the involved Volterra kernel is discrete instead of absolutely continuous, and we have error terms to deal with. The key step in this argument is provided by the following lemma, which may be compared with Lemma 1 of de Bruijn [2].

LEMMA 7. *Let $h^5 \leq u \leq y/2 \log y$, y being sufficiently large, and suppose $\Delta(u) > f^*(u)u^{-1/10}$. Then there exists a number $u_1 \in [u - 1, u]$ for which at least one of the estimates*

$$(5) \quad f^*(u_1 - 1, u_1) \leq \left(1 + O\left(\frac{h^4}{u}\right)\right) \left(f^*(u - 2) - \frac{\Delta(u)}{u^{3/4}}\right)$$

and

$$(5)' \quad f_*(u_1 - 1, u_1) \geq \left(1 + O\left(\frac{h^4}{u}\right)\right) \left(f_*(u - 2) + \frac{\Delta(u)}{u^{3/4}}\right)$$

holds, the implied constants being absolute.

PROOF. Let u satisfy the hypotheses of the lemma and define the sets

$$\begin{aligned} A_+ &= \{u' \in [u - h, u]: f(u') \geq f_*(u) + \tfrac{1}{3}\Delta(u)\}, \\ A_- &= \{u' \in [u - h, u]: f(u') \leq f^*(u) - \tfrac{1}{3}\Delta(u)\}. \end{aligned}$$

Moreover, put

$$\overline{A_\pm} = A_\pm - [0, \eta] = \{u' - t: u' \in A_\pm, 0 \leq t \leq \eta\},$$

where $\eta = (y^{1/10} \log y)^{-1}$.

By Lemma 3 and estimate (iv) of Lemma 4 we have, for $u - h \leq u' \leq u$ and $0 \leq t \leq \eta$, y being sufficiently large,

$$\begin{aligned} \frac{f(u' - t)}{f(u')} &= \frac{\Psi(y^{u' - t})}{\Psi(y^{u'})} \exp\left((\log y) \int_{u' - t}^{u'} \alpha_s ds\right) \\ &= 1 + O(t \log y) + O(y^{-1/3}) \\ &= 1 + O(y^{-1/10}). \end{aligned}$$

In view of our hypotheses on u and $\Delta(u)$, this implies

$$\begin{aligned} f(u' - t) &= f(u') + O(f^*(u)y^{-1/10}) \\ &= f(u') + O(\Delta(u)u^{1/10}y^{-1/10}) \\ &= f(u') + O\left(\frac{\Delta(u)}{(\log y)^{1/10}}\right) \end{aligned}$$

and hence

$$|f(u' - t) - f(u')| \leq \Delta(u)/12,$$

if y is sufficiently large, as we may assume. Thus we have

$$f(u') \geq f_*(u) + \frac{\Delta(u)}{3} - \frac{\Delta(u)}{12} = f_*(u) + \frac{\Delta(u)}{4}$$

for every $u' \in \overline{A_+}$ and

$$f(u') \leq f^*(u) - \Delta(u)/4$$

for every $u' \in \overline{A_-}$.

Applying Lemma 5, we obtain, for $u - 2 \leq u' \leq u$,

$$\begin{aligned} f(u') \left(1 + O\left(\frac{h^3}{u}\right)\right) &= \frac{1}{u' \log y} \sum_{\substack{p^m \leq y^h \\ p \leq y}} \frac{\log p}{p^{m\alpha_{u'}}} f\left(u' - \frac{\log p^m}{\log y}\right) \\ &\leq \max(f^*(u'), f^*(u)) \cdot \frac{1}{u' \log y} \sum_{\substack{p^m \leq y^u \\ p \leq y}} \frac{\log p^m}{p^{m\alpha_{u'}}} \\ &\quad - \frac{\Delta(u)}{4} \cdot \frac{1}{u' \log y} \sum_{\substack{p \leq y \\ u' - (\log p)/(\log y) \in \overline{A_-}}} \frac{\log p}{p^{\alpha_{u'}}}. \end{aligned}$$

By Lemma 6 we have

$$\text{Max}(f^*(u'), f^*(u)) \leq f^*(u-2)(1 + O(h^4/u)),$$

and from part (v) of Lemma 4 we get

$$\frac{1}{u' \log y} \sum_{\substack{p^m \leq y^h \\ p \leq y}} \frac{\log p}{p^{m\alpha_{u'}}} = 1 + O(e^{1-h}) = 1 + O\left(\frac{1}{u}\right).$$

Putting

$$\mu(A, u') = \frac{1}{u' \log y} \sum_{\substack{p \leq y \\ u' - (\log p)/(\log y) \in A}} \frac{\log p}{p^{\alpha_{u'}}} \quad (A \subset \mathbf{R}),$$

we therefore obtain

$$f(u') \left(1 + O\left(\frac{h^3}{u}\right)\right) \leq f^*(u-2) \left(1 + O\left(\frac{h^4}{u}\right)\right) - \frac{\Delta(u)}{4} \mu(\overline{A-}, u').$$

Since

$$\frac{\Delta(u)}{4} \mu(\overline{A-}, u') \leq \frac{\Delta(u)}{4} \leq \frac{f^*(u)}{4} \leq \frac{f^*(u-2)}{4} \left(1 + O\left(\frac{h^4}{u}\right)\right)$$

and, by hypothesis, $u \geq h^5 = (10 \log y)^5$, we can rewrite this as

$$f(u') \leq \left(1 + O\left(\frac{h^4}{u}\right)\right) \left(f^*(u-2) - \frac{\Delta(u)}{4} \mu(\overline{A-}, u')\right).$$

In the same way we get, for $u-2 \leq u' \leq u$,

$$f(u') \geq \left(1 + O\left(\frac{h^4}{u}\right)\right) \left(f_*(u-2) + \frac{\Delta(u)}{4} \mu(\overline{A+}, u')\right).$$

Thus we see that (5) resp. (5)' are implied by

$$(6) \quad \inf\{\mu(\overline{A-}, u') : u_1 - 1 \leq u' \leq u_1\} \geq \frac{4}{u^{3/4}}$$

resp.

$$(6)' \quad \inf\{\mu(\overline{A+}, u') : u_1 - 1 \leq u' \leq u_1\} \geq \frac{4}{u^{3/4}},$$

and to obtain the assertion of Lemma 7, it remains to show, that for some $u_1 \in [u-1, u]$ one of these last two estimates holds. To this end we use the following lemma.

LEMMA 8. *Let A be any set of real numbers and set $\overline{A} = A - [0, \eta]$ with $\eta = y^{-1/10} \log^{-1} y$. Let $h \leq u \leq y/2 \log y$ and suppose $\lambda(\overline{A} \cap [u-1, u-\frac{1}{4}]) \geq \frac{1}{8}$, where λ denotes Lebesgue measure. Then we have $\mu(\overline{A}, u) \geq 4/u^{3/4}$, provided y is sufficiently large (in absolute terms).*

Before proving this result, we show how it implies Lemma 7.

Suppose (6) does not hold for $u_1 = u - \frac{1}{2}$ and let $u_0 \in [u - \frac{3}{2}, u - \frac{1}{2}]$ be such that $\mu(\overline{A-}, u_0) < 4/u^{3/4} \leq 4/u_0^{3/4}$. We shall show that then (6)' holds for $u_1 = u_0 + \frac{1}{2} \subset [u-1, u]$. As we remarked above, this is sufficient for the proof of Lemma 7.

Lemma 8 with $A = \overline{A_-}$ and u replaced by u_0 implies, in view of our hypothesis $\mu(\overline{A_-}, u_0) < 4u_0^{-3/4}$, $\lambda(\overline{A_-} \cap [u_0 - 1, u_0 - \frac{1}{4}]) < \frac{1}{8}$. Since, by the definition of $\overline{A_{\pm}}$,

$$\overline{A_+} \cup \overline{A_-} \supset A_+ \cup A_- \supset [u - h, u] \supset [u_0 - 1, u_0 - \frac{1}{4}],$$

we conclude

$$\lambda(\overline{A_+} \cap [u_0 - 1, u_0 - \frac{1}{4}]) \geq \frac{3}{4} - \frac{1}{8} = \frac{5}{8}.$$

But then we have

$$\lambda(\overline{A_+} \cap [u' - 1, u' - \frac{1}{4}]) \geq \frac{1}{8}$$

for every $u' \in [u_0 - \frac{1}{2}, u_0 + \frac{1}{2}]$, and a second application of Lemma 8 now yields (6)' for $u_1 = u_0 + \frac{1}{2}$, as wanted.

PROOF OF LEMMA 8. Let A and u satisfy the hypotheses of the lemma. It is easy to see that \overline{A} can be represented as a union of disjoint intervals of length $\geq \eta$ and $\leq 2\eta$. Let I be such an interval with endpoints t_1 and t_2 , and suppose $I \subset [u - 1, u - \frac{1}{4}]$, so that $u - 1 \leq t_1 < t_2 \leq u - \frac{1}{4}$, $\eta \leq t_2 - t_1 \leq 2\eta$. We then have

$$\begin{aligned} \sum_{u - (\log p)/(\log y) \in I} \log p &\geq \sum_{y^{u-t_2} < p < y^{u-t_1}} \log p \\ &= (y^{u-t_1} - y^{u-t_2}) \left(1 + O\left(\frac{1}{\log y}\right)\right) \end{aligned}$$

by Hoheisel's prime number theorem in the form, due to Heath-Brown [5],

$$\sum_{x < p \leq x+z} \log p = z \left(1 + O\left(\frac{1}{\log x}\right)\right) \quad (x \geq z \geq x^{7/12})$$

and the inequality

$$\begin{aligned} y^{u-t_1} - y^{u-t_2} &\geq y^{u-t_2}(y^\eta - 1) \geq y^{u-t_2}\eta \log y \\ &= y^{u-t_2-1/10} \geq y^{(u-t_2)(1-4/10)} \geq y^{(u-t_2)7/12}. \end{aligned}$$

Therefore we get, uniformly for $\alpha \in [0, 1]$,

$$\begin{aligned} \sum_{\substack{p \leq y \\ u - (\log p)/(\log y) \in I}} \frac{\log p}{p^\alpha} &= \frac{1}{y^{(u-t_1)\alpha}} (y^{u-t_1} - y^{u-t_2}) \left(1 + O\left(\frac{1}{\log y}\right)\right) \\ &= \left(1 + O\left(\frac{1}{\log y}\right)\right) \int_{y^{u-t_2}}^{y^{u-t_1}} \frac{ds}{s^\alpha} = (\log y + O(1)) \int_{t_1}^{t_2} y^{(u-t)(1-\alpha)} dt \\ &= (\log y + O(1)) \int_I y^{(u-t)(1-\alpha)} dt. \end{aligned}$$

Since the intervals I of the type described above are all contained in the set $\overline{A} \cap [u - 1, u - \frac{1}{4}]$, and form a disjoint covering of the set $\overline{A} \cap [u - 1 + \eta, u - \frac{1}{4} - \eta]$, we deduce

$$\begin{aligned} \sum_{\substack{p \leq y \\ u - (\log p)/(\log y) \in \overline{A} \cap [u-1, u-1/4]}} \frac{\log p}{p^\alpha} &\geq (\log y + O(1)) \int_{\overline{A} \cap [u-1+\eta, u-1/4-\eta]} y^{(u-t)(1-\alpha)} dy \\ &\geq (\log y + O(1)) y^{(1-\alpha)/4} (\lambda(\overline{A} \cap [u - 1, u - 1/4]) - 2\eta) \\ &\geq \frac{1}{8} (\log y + O(1)) y^{(1-\alpha)/4}. \end{aligned}$$

Now, according to Lemma 4(iv), we have $\alpha_u \in [0, 1]$, if y is sufficiently large and $u \in [h, y/2 \log y]$, as we have assumed. Thus we can apply the above estimate with $\alpha = \alpha_u$ and obtain

$$\begin{aligned} \mu\left(\bar{A} \cap \left[u-1, u-\frac{1}{4}\right], u\right) &= \frac{1}{u \log y} \sum_{\substack{p \leq y \\ u-(\log p)/(\log y) \in \bar{A} \cap [u-1, u-1/4]}} \frac{\log p}{p^{\alpha_u}} \\ &\geq \left(\frac{1}{8} + O\left(\frac{1}{\log y}\right)\right) \frac{y^{(1-\alpha_u)/4}}{u}. \end{aligned}$$

Since, by estimates (ii) and (iv) of Lemma 4 and our hypothesis $u \geq h = 10 \log y$,

$$\begin{aligned} y^{(1-\alpha_u)/4} &\gg e^{\xi_u/4} \geq (u \xi_u)^{1/4} \gg (u \log u)^{1/4} \\ &\gg u^{1/4} (\log \log y)^{1/4}, \end{aligned}$$

the desired inequality

$$\mu(\bar{A} \cap [u-1, u-1/4], u) \geq 4/u^{3/4}$$

follows, provided y is sufficiently large.

This completes the proof of Lemma 8.

6. Proof of Theorem 2. Conclusion. Theorem 2 will be an easy consequence of Theorem 3 and the following

LEMMA 9. *Let $h^{100} \leq u \leq y/\log 2$, y being sufficiently large, and suppose $\Delta(u-h) \geq f^*(u-h)(u-h)^{-1/10}$. Then we have*

$$\Delta(u) \leq \Delta(u-2h)(1-u^{-5/6}).$$

PROOF. Let u satisfy the hypotheses of the lemma. Applying Lemma 7 with $u-h$ in place of u , we see that for a suitable $u_1 \in [u-h-1, u-h]$ at least one of the estimates

$$(7) \quad f^*(u_1-1, u_1) \leq \left(1 + O\left(\frac{h^4}{u}\right)\right) \left(f^*(u-h-2) - \frac{\Delta(u-h)}{u^{3/4}}\right)$$

and

$$(7)' \quad f_*(u_1-1, u_1) \geq \left(1 + O\left(\frac{h^4}{u}\right)\right) \left(f_*(u-h-2) + \frac{\Delta(u-h)}{u^{3/4}}\right)$$

holds. We shall show that (7) resp. (7)' imply (for sufficiently large y)

$$(8) \quad f^*(u) \leq \left(1 + O\left(\frac{h^5}{u}\right)\right) \left(f^*(u-2h) - \frac{\Delta(u)}{u^{4/5}}\right)$$

resp.

$$(8)' \quad f_*(u) \geq \left(1 + O\left(\frac{h^5}{u}\right)\right) \left(f_*(u-2h) + \frac{\Delta(u)}{u^{4/5}}\right).$$

Either of these two last estimates yields the assertion of the lemma. Suppose, for example, that (8) holds. By means of Lemma 6, we get

$$\begin{aligned} \Delta(u) &= f^*(u) - f_*(u) \\ &\leq \left(1 + O\left(\frac{h^6}{u}\right)\right) f^*(u-2h) - \frac{\Delta(u)}{u^{4/5}} - f_*(u-2h) \\ &= \Delta(u-2h) - \frac{\Delta(u)}{u^{4/5}} + O\left(\frac{f^*(u-2h)h^5}{u}\right). \end{aligned}$$

Since

$$\begin{aligned} f^*(u-2h) &\leq f^*(u-h) + \Delta(u-2h) \\ &\leq \Delta(u-h)(u-h)^{1/10} + \Delta(u-2h) = O(\Delta(u-2h)u^{1/10}), \end{aligned}$$

the desired inequality

$$\Delta(u) \leq \Delta(u-2h) \left(1 - u^{-5/6}\right)$$

follows, provided $u \geq h^{100}$ and y is sufficiently large, as we have assumed in the lemma. A similar argument applies in the case (8)' holds.

It therefore remains to prove the implications (7) \Rightarrow (8) and (7)' \Rightarrow (8)'. We shall only prove the first one; the proof of the second is similar.

Suppose that (7) holds for some $u_1 \in [u-h-1, u-h]$. Put, for $n \geq 1$, $u_n = u_1 + (n-1)/2$, and define Δ_n by

$$f^*(u_1-1, u_n) = f^*(u-2h) - \Delta_n.$$

Let $N = [2h+1]$. We shall show (for sufficiently large y)

$$(9) \quad \Delta_N \geq \frac{\Delta(u)}{u^{4/5}} + O\left(\frac{f^*(u-2h)h^5}{u}\right).$$

Since, by the definition of u_N ,

$$u_1 + h - \frac{1}{2} \leq u_N \leq u_1 + h \leq u \leq u_1 + h + 1,$$

we get from Lemma 6

$$\begin{aligned} f^*(u) &\leq f^*(u_N) \left(1 + O\left(\frac{h^4}{u}\right)\right) \leq f^*(u_1-1, u_N) \left(1 + O\left(\frac{h^4}{u}\right)\right) \\ &= (f^*(u-2h) - \Delta_N) \left(1 + O\left(\frac{h^4}{u}\right)\right). \end{aligned}$$

Thus we see that (8) follows from (9), and it suffices to prove the latter estimate.

By Lemma 6 and the inequality $h+2 \leq 2h$, we have

$$f^*(u-h-2) \leq f^*(u-2h)(1 + O(h^4/u))$$

and

$$\Delta(u-h) \geq \Delta(u) + O(f^*(u-2h)h^4/u).$$

Hence (7) implies

$$\begin{aligned} \Delta_1 &= f^*(u-2h) - f^*(u_1-1, u_1) \\ &\geq \frac{\Delta(u-h)}{u^{3/4}} + O\left(\frac{f^*(u-2h)h^5}{u}\right) \\ &\geq \frac{\Delta(u)}{u^{3/4}} + O\left(\frac{f^*(u-2h)h^5}{u}\right). \end{aligned}$$

We shall show

$$(10) \quad \Delta_{n+1} \geq \Delta_n \left(1 - \left(\frac{9}{10}\right)^n\right) + O\left(\frac{f^*(u-2h)h^4}{u}\right)$$

uniformly for $1 \leq n \leq N-1$. This together with the above estimate for Δ_1 yields

$$\begin{aligned}\Delta_N &\geq \Delta_1 \prod_{n \geq 1} \left(1 - \left(\frac{9}{10}\right)^n\right) + O\left(N \cdot \frac{f^*(u-2h)h^4}{u}\right) \\ &\geq \frac{\Delta(u)}{u^{3/4}} \prod_{n \geq 1} \left(1 - \left(\frac{9}{10}\right)^n\right) + O\left(\frac{f^*(u-2h)h^5}{u}\right)\end{aligned}$$

and hence (9), if y and therefore $u (\geq h^{100} = (10 \log y)^{100})$ are sufficiently large.

Let $1 \leq n \leq N-1$ be fixed. For the proof of (10) we may suppose $\Delta_{n+1} < \Delta_n$, i.e.

$$f^*(u-1, u_{n+1}) > f^*(u_1-1, u_n).$$

This means that for some $u' \in (u_n, u_{n+1}]$ we have

$$f^*(u_1-1, u_{n+1}) \leq f(u')(1+1/u).$$

By Lemma 5 and the definition of Δ_n we have

$$\begin{aligned}f^*(u-2h) - \Delta_{n+1} &= f^*(u_1-1, u_{n+1}) \leq f(u')(1+1/u) \\ &= \left(1 + O\left(\frac{h^3}{u}\right)\right) \frac{1}{u' \log y} \sum_{\substack{p^m \leq y^h \\ p \leq y}} \frac{\log p}{p^{m\alpha_{u'}}} f\left(u' - \frac{\log p^m}{\log y}\right) \\ &\leq \left(1 + O\left(\frac{h^3}{u}\right)\right) \{f^*(u_1-1, u_{n+1})\lambda_1 + f^*(u_1-1, u_n)\lambda_2 + f^*(u_1-1)\lambda_3\} \\ &\leq \left(1 + O\left(\frac{h^3}{u}\right)\right) (\lambda_1 + \lambda_2 + \lambda_3) \max(f^*(u-2h), f^*(u_1-1)) - \lambda_1 \Delta_{n+1} - \lambda_2 \Delta_n,\end{aligned}$$

where

$$\begin{aligned}\lambda_1 &= \frac{1}{u' \log y} \sum_{\substack{p^m \leq y^{u'-u_n} \\ p \leq y}} \frac{\log p}{p^{m\alpha_{u'}}}, \\ \lambda_2 &= \frac{1}{u' \log y} \sum_{\substack{y^{u'-u_n} < p^m \leq y^{u'-(u_1-1)} \\ p \leq y}} \frac{\log p}{p^{m\alpha_{u'}}}\end{aligned}$$

and

$$\lambda_3 = \frac{1}{u' \log y} \sum_{\substack{y^{u'-(u_1-1)} < p^m \leq y^h \\ p \leq y}} \frac{\log p}{p^{m\alpha_{u'}}}.$$

Since, by Lemma 6,

$$\max(f^*(u-2h), f^*(u_1-1)) \leq \left(1 + O\left(\frac{h^4}{u}\right)\right) f^*(u-2h)$$

and since

$$\begin{aligned}\lambda_1 + \lambda_2 + \lambda_3 &= \frac{1}{u' \log y} \sum_{\substack{p^m \leq y^h \\ p \leq y}} \frac{\log p^m}{p^{m\alpha_{u'}}} \\ &\leq \frac{1}{u' \log y} \sum_{p \leq y} \frac{\log p}{p^{\alpha_{u'}} - 1} = 1,\end{aligned}$$

we deduce

$$f^*(u - 2h) - \Delta_{n+1} \leq \left(1 + O\left(\frac{h^4}{u}\right)\right) f^*(u - 2h) - \lambda_1 \Delta_{n+1} - \lambda_2 \Delta_n$$

and hence

$$\Delta_{n+1} \geq \frac{\lambda_2}{1 - \lambda_1} \Delta_n + O\left(\frac{f^*(u - 2h)h^4}{u(1 - \lambda_1)}\right).$$

Thus, to obtain (10), it suffices to show

$$(11) \quad 1 - \lambda_1 \gg 1, \quad \frac{\lambda_2}{1 - \lambda_1} \geq 1 - \left(\frac{9}{10}\right)^n.$$

Using estimates (iv) of Lemma 4 and the inequality $u' - u_n \leq u_{n+1} - u_n \leq \frac{1}{2}$, we get, for sufficiently large y ,

$$\begin{aligned} 1 - \lambda_1 &\geq 1 - \frac{1}{u' \log y} \sum_{\substack{p^m \leq \sqrt{y} \\ p \leq y}} \frac{\log p}{p^{m\alpha_{u'}}} \\ &= \frac{1}{u' \log y} \sum_{\substack{p^m > \sqrt{y} \\ p \leq y}} \frac{\log p}{p^{m\alpha_{u'}}} \gg \frac{1}{u' \log y} \sum_{p \leq y} \frac{\log p}{p^{\alpha_{u'} - 1}} = 1, \end{aligned}$$

i.e. the first part of (11). To prove the second part, we first note that for every fixed $p \leq y$ and $t \geq 0$ we have

$$\sum_{\substack{m \geq 1 \\ p^m > y^t}} \frac{1}{p^{m\alpha_{u'}}} = \frac{1}{p^{m_0(t)\alpha_{u'}}(1 - p^{-\alpha_{u'}})}$$

with

$$m_0(t) = \left\lceil \frac{t \log y}{\log p} \right\rceil + 1.$$

It follows that, if $t \geq 1$, then

$$\begin{aligned} \sum_{\substack{m \geq 1 \\ p^m > y^t}} \frac{1}{p^{m\alpha_{u'}}} &= p^{(m_0(1/2) - m_0(t))\alpha_{u'}} \sum_{\substack{m \geq 1 \\ p^m > \sqrt{y}}} \frac{1}{p^{m\alpha_{u'}}} \\ &\leq \min\left(y^{(1-t)\alpha_{u'}}, y^{-\alpha_{u'}/4}\right) \sum_{\substack{m \geq 1 \\ p^m > \sqrt{y}}} \frac{1}{p^{m\alpha_{u'}}} \end{aligned}$$

In view of the inequalities $u' - u_n \leq 1/2$ and $u' - u_1 + 1 \geq (n + 1)/2 \geq 1$, we therefore get

$$\begin{aligned} \frac{\lambda_2}{1 - \lambda_1} &= \left(\sum_{\substack{y^{u' - u_n} < p^m \leq y^{u' - u_1 + 1} \\ p \leq y}} \frac{\log p}{p^{m\alpha_{u'}}} \right) \left(\sum_{\substack{p^m > y^{u' - u_n} \\ p \leq y}} \frac{\log p}{p^{m\alpha_{u'}}} \right)^{-1} \\ &= 1 - \left(\sum_{p \leq y} \log p \sum_{\substack{m \geq 1 \\ p^m > y^{u' - u_1 + 1}}} \frac{1}{p^{m\alpha_{u'}}} \right) \left(\sum_{p \leq y} \log p \sum_{\substack{m \geq 1 \\ p^m > y^{u' - u_n}}} \frac{1}{p^{m\alpha_{u'}}} \right)^{-1} \\ &\geq 1 - \min\left(y^{(1-n)\alpha_{u'}/2}, y^{-\alpha_{u'}/4}\right). \end{aligned}$$

Since, by part (iv) of Lemma 4, $\alpha_{u'} \geq 1/(\log y)$ for sufficiently large y , we have

$$\min \left(y^{(1-n)\alpha_{u'}/2}, y^{-\alpha_{u'}/4} \right) \leq \min \left(\frac{1}{e^{(n-1)/2}}, \frac{1}{e^{1/4}} \right) \leq \left(\frac{9}{10} \right)^n$$

and hence

$$\frac{\lambda_2}{1 - \lambda_1} \geq 1 - \left(\frac{9}{10} \right)^n.$$

Thus (11) is proved and the proof of Lemma 9 is complete.

PROOF OF THEOREM 2. Putting $x = y^u$, $c = y^t$, we have to show that

$$\frac{\Psi(y^u, y)}{\Psi(y^{u-t}, y)} = y^{t\alpha_{u-t}} \left(1 + O \left(\frac{1}{u^{1/10}} \right) \right)$$

holds uniformly for $0 \leq t \leq 1$ and $1 \leq u - t \leq y/4 \log y$. We shall prove this for the slightly larger range $1 \leq u \leq y/2 \log y$, $0 \leq t \leq \min(u - t, 1)$.

As we mentioned already, we may assume, without loss of generality, y to be sufficiently large. Moreover, if $1 \leq u \leq \exp \sqrt{h} = \exp \sqrt{10 \log y}$, then the asserted estimate is an immediate consequence of (1) (which is valid in this range) and the estimate

$$\frac{\rho(u)}{\rho(u-t)} = e^{-t\xi_u} \left(1 + O \left(\frac{1}{u} \right) \right) = y^{t(\alpha_{u-t}-1)} \left(1 + O \left(\frac{1}{u} \right) \right),$$

which follows from [1, Lemma 3] and Lemma 4. It remains therefore to treat the case $\exp \sqrt{h} \leq u \leq y/2 \log y$ with y sufficiently large.

By the definition of $f(u)$ and part (iii) of Lemma 4, we have

$$\begin{aligned} \frac{\Psi(y^u, y)}{\Psi(y^{u-t}, y)} &= \frac{f(u)}{f(u-t)} \exp \left\{ (\log y) \int_{u-t}^u \alpha_s ds \right\} \\ &= \frac{f(u)}{f(u-t)} y^{t\alpha_{u-t}} \left(1 + O \left(\frac{\log y}{u} \right) \right) \\ &= \frac{f(u)}{f(u-t)} y^{t\alpha_{u-t}} \left(1 + O \left(\frac{(\log u)^2}{u} \right) \right). \end{aligned}$$

Since

$$\left| \frac{f(u)}{f(u-t)} - 1 \right| \leq \frac{f^*(u) - f_*(u)}{f(u-t)} \leq \frac{\Delta(u)}{f^*(u) - \Delta(u)},$$

the result follows, if we can show

$$(12) \quad \Delta(u) = O(f^*(u)u^{-1/10})$$

for $\exp \sqrt{h} \leq u \leq y/2 \log y$.

To prove (12), let $u \in [e^{\sqrt{h}}, y/2 \log y]$ be fixed and put, for $n \geq 0$, $u_n = u - nh$. Let $N = 2[u^{6/7}]$ and note that by our hypotheses on u and y

$$u_n \geq u_N \geq u - 2u^{6/7} \geq h^{100} \quad (0 \leq n \leq N).$$

If

$$\Delta(u_n) \geq f^*(u_n)u_n^{-1/10} \quad (1 \leq n \leq N),$$

we get from Lemma 9

$$\Delta(u_{2n}) \leq \Delta(u_{2(n+1)}) \left(1 - \frac{1}{u_{2n}^{5/6}} \right) \leq \Delta(u_{2(n+1)}) \exp \left(-\frac{1}{u_{2n}^{5/6}} \right) \quad (0 \leq n < N/2)$$

and hence

$$\Delta(u) = \Delta(u_0) \leq \Delta(u_N) \exp\left(-\frac{N/2-1}{u^{5/6}}\right) \ll \Delta(u_N) \exp(-u^{1/42}).$$

This implies (12), since by Theorem 3 and our hypothesis $u \geq e^{\sqrt{h}}$,

$$\Delta(u_N) \leq f^*(u_N) = \exp(O(h^5)) = f^*(u) \exp(O(h^5)) = f^*(u) \exp(O((\log u)^{10})).$$

Suppose now that, for some $n \in \{1, \dots, N\}$

$$\Delta(u_n) < f^*(u_n) u_n^{-1/10}$$

holds. Applying Lemma 6 repeatedly, we see that

$$\begin{aligned} \Delta(u) &= \Delta(u_0) = f^*(u_0) - f_*(u_0) \leq f^*(u_n) \left(1 + O\left(\frac{nh^4}{u}\right)\right) - f_*(u_n) \\ &= \Delta(u_n) + O\left(\frac{f^*(u_n)h^4N}{u}\right) = O\left(\frac{f^*(u_n)}{u^{1/10}}\right) \end{aligned}$$

and further

$$\begin{aligned} f^*(u_n) &= f_*(u_n) + \Delta(u_n) \ll f_*(u_n) \leq f_*(u_0)(1 + O(h^4N/u)) \\ &\ll f_*(u_0) = f_*(u) \leq f^*(u). \end{aligned}$$

Thus we obtain again (12).

The proof of Theorem 2 is now complete.

NOTE ADDED IN PROOF. This work was completed in March 1984. Since then, G. Tenenbaum and the present author, using a different, analytic approach, have obtained an asymptotic formula for $\Psi(x, y)$, valid uniformly in $x \geq y \geq 2$. It follows from this formula that the estimate of Theorem 2 holds in the stronger form

$$\Psi(c, x, y) = c^{\alpha(x, y)} \Psi(x, y) \left(1 + O\left(\frac{\log y}{\log x}\right) + O\left(\frac{\log y}{y}\right)\right)$$

uniformly for $x \geq y \geq 2$ and $1 \leq c \leq y$. These results appear in a paper entitled *On integers free of large prime factors*, Trans. Amer. Math. Soc. **296** (1986), 265–290.

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